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Determinants on lens spaces and cyclotomic units

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Abstract

The Laplacian functional determinants for conformal scalars and coexact one-forms are evaluated in closed form on inhomogeneous lens spaces of certain orders, including all odd primes when the essential part of the expression is given, formally, as a cyclotomic unit.

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1. Introduction

Given the solution of a spectral problem, for some differential operator say, the calculation of the corresponding functional determinant could be regarded as just a computational challenge but there are, of course, uses for such objects. In physics they determine the one-loop effective action. In mathematics, for the de Rham complex, they occur in the analytic torsion, and elsewhere.

It is not necessary to have the spectrum explicitly available in order to calculate the determinant, but it helps. For this reason many discussions revolve around exactly solvable cases and prominent amongst these are the spheres. Some relevant brief history was attempted in [1] so nothing more will be said on this just now.

The intention of this paper is to present a small contribution to the general store of knowledge about spherical determinants, in particular on lens spaces. These have played an important part in discussions of analytic torsion [2].

In an earlier work [3], amongst other things, I calculated the determinants on the even homogeneous lens spaces, S^3/\mathbb{Z}_q , (see also [4]). In this work I turn to the inhomogeneous case. Curiously the results turn out to look quite different when q is an odd prime. A variety of approaches is offered and I also make some further technical analysis of the homogeneous case.

2. Lens space spectrum

Thinking of the lens space as the spatial section of an Einstein universe, I consider only the conformal scalar and divergenceless Maxwell vector (coexact one-form) eigenproblems in

computational detail. There is a lot of prior mathematical work on the spectral problem, as it is relevant for the analytic torsion (the minimal scalar is needed here) and the η invariant, but I shall take the spectral details as given in the earlier works [5–7] since they are in the form which I wish to use.

The differential operators under consideration are the scalar Laplacian, with an addition to make it conformally invariant (in four dimensions), the de Rham Laplacian, and, for spin-half, the square of the Dirac operator.

The fact that the spectrum is composed of squares of integers (up to a scaling) means that the number theory is almost bound to appear somewhere in the story, and this will happen.

The eigenfunctions are labelled by two angular momenta, L and J where $J = L + j$ and j is the spin of the field ($j = 0, 1/2, 1$). The massless polarization conditions produce the restriction $J = L \pm j$, making L ultimately a sufficient label.

The spectrum on the full sphere of radius a is

$$\begin{aligned}\lambda_n^+ &= \frac{1}{a^2}(j+n)^2, & n = 1, 2, \dots, & \quad (J = L + j) \\ \lambda_n^- &= \frac{1}{a^2}(j-n)^2, & n = 2j+1, 2j+2, \dots, & \quad (J = L - j)\end{aligned}\tag{1}$$

where $n = 2L + 1 \in \mathbb{Z}$ and the spectrum has been split into the parts that arise from the positive and the negative spectra of the corresponding first order (pseudo)-operators. If these parts are to be united, it is necessary to distinguish $j = 0$ from $j > 0$. The positive and negative eigenfunctions are related by the interchange, $L \leftrightarrow J$, a parity transformation. The point is that, if $j = 0$, these modes are degenerate and must not be counted twice. Hence it is necessary to introduce a degeneracy factor, $d(j) = 1/2$, $j = 0$ while $d(j) = 1$, $j > 0$.

The factoring, S^3/Γ , does not alter the positive–negative split and the eigenvalues are still as in (1) except that the range of n is modified. The degeneracy will contain this information.

The deck group, Γ , has left and right actions with typical elements, γ_L and γ_R . The degeneracy takes the $SO(4)$ character form [7]

$$d(L, J) = \frac{1}{|\Gamma|} \sum_{\gamma_L, \gamma_R} \chi^{(L)}(\gamma_L) \chi^{(J)}(\gamma_R),\tag{2}$$

in terms of the $SU(2)$ characters,

$$\chi^{(L)}(\gamma) = \frac{\sin(2L+1)\theta_\gamma}{\sin \theta_\gamma},$$

where θ_γ is the ‘radial’ angular coordinate labelling the group element, γ .

The freedom allowed by the non-trivial fundamental group to twist the field by a representation, $\text{Hom}(\Gamma, U(1))$, has not been incorporated here. This would be necessary if the torsion was under consideration.

The spectral data are now combined into a ζ -function on S^3/Γ , cf [5], by adding the positive and negative parts (subtraction would give the η invariant).

Trigonometry gives

$$\zeta_3(s) = \frac{a^{2s} d(j)}{|\Gamma|} \sum_{\alpha, \beta} \frac{2}{\cos \beta - \cos \alpha} \sum_{n=j}^{\infty} \frac{1}{n^{2s}} (\cos n\beta \cos j\alpha - \cos n\alpha \cos j\beta),\tag{3}$$

where

$$\alpha = \theta_R + \theta_L, \quad \beta = \theta_R - \theta_L.$$

From this point on, the expressions do not apply to spin-half because the summation variable has been shifted by j to reach (3).

Initially, I will proceed without placing the analysis in a wider context, i.e. just from a pedestrian calculational viewpoint.

As in [5], the ζ -function is written in terms of the simplest Epstein ζ -function, defined by

$$Z \left| \begin{matrix} g \\ h \end{matrix} \right| (s) = \sum_{m=-\infty}^{\infty} |m + g|^{-s} e^{2\pi imh}, \tag{4}$$

with the understanding that, if $g = 0$, the $m = 0$ term is omitted. Then,

$$\zeta_3(s) = \frac{a^{2s} d(j)}{|\Gamma|} \sum_{\alpha, \beta} \frac{1}{\cos \beta - \cos \alpha} \left(\cos j\alpha Z \left| \begin{matrix} 0 \\ \beta/2\pi \end{matrix} \right| (2s) - \cos j\beta Z \left| \begin{matrix} 0 \\ \alpha/2\pi \end{matrix} \right| (2s) \right). \tag{5}$$

A remark of possible interest is that, in our previous work, [3], the ζ -function on one-sided (homogeneous) lens spaces was reduced to a ζ -function on a factored *two*-sphere. In the case under study here, the Epstein function (4) is a twisted ζ -function on the *one*-sphere, if $g = 0$.

I also note that $\zeta_3(0) = 0$ for spin-0 but $\zeta_3(0) = 1$ for spin-1. This will come up later.

The advantage of the Epstein expression is the existence of a functional relation that allows (5) to be replaced by the image form,

$$\zeta_3(s) = \frac{a^{2s} d(j)}{|\Gamma|} \pi^{2s-1/2} \frac{\Gamma(1/2 - s)}{\Gamma(s)} \times \sum_{\alpha, \beta} \frac{1}{\cos \beta - \cos \alpha} \left(\cos j\alpha Z \left| \begin{matrix} \beta/2\pi \\ 0 \end{matrix} \right| (1 - 2s) - \cos j\beta Z \left| \begin{matrix} \alpha/2\pi \\ 0 \end{matrix} \right| (1 - 2s) \right). \tag{6}$$

So far everything has been for a general Γ . The case of lens spaces, $L(q; \lambda_1, \lambda_2)$, is covered by the choice of angles

$$\frac{\alpha}{2\pi} = \frac{pv_1}{q}, \quad \frac{\beta}{2\pi} = \frac{pv_2}{q}, \tag{7}$$

where $p, = 0, \dots, q - 1$, labels γ . v_1 and v_2 are integers coprime to q , with λ_1 and λ_2 their mod q inverses.

By an appropriate selection of a q th root of unity, it is possible to set $v_1 = 1$, i.e. $\lambda_1 = 1$, without loss of generality. Any pair, (v_1, v_2) , can be reduced to $(1, v)$ by multiplying through by the mod q inverse of v_1 .

The simple, one-sided lens space, $L(q; 1, 1)$, corresponds to setting $v = 1$ so that $\theta_L = 0, \theta_R = 2\pi p/q$ and the ζ -function becomes a derivative of an Epstein function. This is not the case I am interested in just now. Indeed, if the method to be presented later is to run smoothly, it is necessary that the denominator, $\cos \beta - \cos \alpha$, should never be zero. This puts conditions on q and v which can be simply, but not uniquely, satisfied by choosing q to be odd prime, when all v from 2 to $q - 2$ are covered. I will do this from now on for organizational convenience.

The $p = 0$ value corresponds to the identity element of Γ , and is best separated. For the two spins, the identity ζ -functions are

$$\begin{aligned} \zeta_3^{id}(s) &= a^{2s} \frac{1}{|\Gamma|} \zeta_R(2s - 2), & j = 0 \\ \zeta_3^{id}(s) &= a^{2s} \frac{2}{|\Gamma|} (\zeta_R(2s - 2) - \zeta_R(2s)), & j = 1. \end{aligned} \tag{8}$$

These differ from the full sphere expressions only by the $1/|\Gamma|$ volume factor.

3. Lens space determinants

On differentiating (5) at $s = 0$ one encounters the right-hand side bracket evaluated at a point where the Z have a pole. For spin-0 these cancel, but not for spin-1 when they will combine with the $1/\Gamma(s)$. There is no problem with this, but I proceed in an alternative fashion eliminating the pole using a limiting procedure.

If $g \neq 0$, the Z of (4) has no pole at $s = 1$. So I insert a non-zero g and let $g \rightarrow 0$ near the end after the differentiation with respect to s . It is necessary to allow for the extra term at $m = 0$ introduced in this way. Hence, instead of (6),

$$\begin{aligned} \zeta_3(s) &= \frac{a^{2s} d(j)}{|\Gamma|} \pi^{2s-1/2} \frac{\Gamma(1/2-s)}{\Gamma(s)} \sum_{\alpha, \beta} \frac{1}{\cos \beta - \cos \alpha} \\ &\quad \times \lim_{g \rightarrow 0} \left(\cos j\alpha Z \left| \frac{\beta/2\pi}{g} \right| (1-2s) - \cos j\beta Z \left| \frac{\alpha/2\pi}{g} \right| (1-2s) \right) \\ &\quad + \frac{d(j)}{|\Gamma|} \sum_{\alpha, \beta} \frac{\cos j\beta - \cos j\alpha}{\cos \beta - \cos \alpha} \lim_{g \rightarrow 0} \left(\frac{a}{g} \right)^{2s}. \end{aligned} \quad (9)$$

Now I can employ a formula of Epstein's, valid for ν and q integral and coprime [8],

$$Z \left| \frac{\nu/q}{g} \right| (1) = -2 \sum_{k=0}^{q-1} e^{-2\pi i(k+g)\nu/q} \log \sin(\pi(g+k)/q), \quad (10)$$

to give

$$\lim_{g \rightarrow 0} Z \left| \frac{\nu/q}{g} \right| (1) = -2 \sum_{k=1}^{q-1} e^{-2\pi i k \nu/q} \log \sin(\pi k/q) - 2 \lim_{g \rightarrow 0} \log \sin(\pi g/q), \quad (11)$$

which is a real quantity and so the exponential can be replaced by a cosine (set $k \rightarrow q - k$).

I note that Epstein's derivation of (10) assumes only the standard summation,

$$Z \left| \frac{0}{h} \right| (1) = -2 \log(2 \sin \pi h), \quad (12)$$

and also that it can be used to streamline some of Ray's algebra [2].

Since the bracket on the right-hand side of (6) is now finite at $s = 0$, in evaluating the derivative at 0, $\zeta_3'(0)$, one needs differentiate only the $1/\Gamma(s)$ factor,

$$\begin{aligned} \zeta_3'(0) &= -2 \frac{d(j)}{q} \sum_{p=1}^{q-1} \sum_{k=1}^{q-1} \frac{\cos \frac{2\pi j\nu p}{q} \cos \frac{2\pi p k}{q} - \cos \frac{2\pi j p}{q} \cos \frac{2\pi \nu p k}{q}}{\cos \frac{2\pi p}{q} - \cos \frac{2\pi \nu p}{q}} \log \sin \frac{\pi k}{q} \\ &\quad + 2 \frac{d(j)}{q} \sum_{p=1}^{q-1} \frac{\cos \frac{2\pi j p}{q} - \cos \frac{2\pi j \nu p}{q}}{\cos \frac{2\pi p}{q} - \cos \frac{2\pi \nu p}{q}} \log \frac{\pi a}{q} + \zeta_3^{id'}(0). \end{aligned} \quad (13)$$

The second term on the right is the residual effect of the 'zero mode' for spin-1 and the last term is the contribution of the identity element from (8).

The appearance of the radius, a , in the spin-1 expression reflects the non-vanishing of $\zeta_3(0)$ and the resulting scaling dependence.

I write (13) cosmetically as, after a few cancellations,

$$\begin{aligned} \zeta'_3(0) &= -\sum_{k=1}^{q-1} A_k(0) \log \sin \frac{\pi k}{q} + \frac{2}{q} \zeta'_R(-2), \quad j = 0 \\ \zeta'_3(0) &= -\sum_{k=1}^{q-1} A_k(1) \log \sin \frac{\pi k}{q} + \frac{4}{q} \zeta'_R(-2) + 2 \log(2\pi a) - \frac{2(q-1)}{q} \log 2q, \quad j = 1 \end{aligned} \tag{14}$$

where the (rational) coefficients, $A_k(j)$, are defined by

$$A_k(j) = 2 \frac{d(j)}{q} \sum_{p=1}^{q-1} \frac{\cos \frac{2\pi j \nu p}{q} \cos \frac{2\pi p k}{q} - \cos \frac{2\pi j p}{q} \cos \frac{2\pi \nu p k}{q}}{\cos \frac{2\pi p}{q} - \cos \frac{2\pi \nu p}{q}}. \tag{15}$$

I also remark that the same result, (13), follows without a special limit by using the remainder at the pole in the Epstein ζ -function. This is a dilogarithm, ψ , function, an equivalent statement being,

$$Z' \left| \begin{matrix} 0 \\ h \end{matrix} \right| (0) = -\gamma - \log 2\pi - \psi \left(\frac{h}{2\pi} \right) - \psi \left(1 - \frac{h}{2\pi} \right), \tag{16}$$

which could be used directly in (5).

One can then employ Gauss' famous formula for $\psi(p/q)$, or better, a formula that appears during a proof of this relation, [9] p 146, which I reproduce (see also [10] p 13),

$$\psi \left(\frac{p}{q} \right) + \psi \left(1 - \frac{p}{q} \right) = -2\gamma - 2 \log 2 + \sum_{k=1}^{q-1} \cos \left(\frac{2\pi p k}{q} \right) \log 2 \sin \frac{\pi q}{k}. \tag{17}$$

One could also take the attitude that using the Epstein equation, (10) provides a neat proof of (17), and hence of Gauss' formula.

The functional determinants are conventionally defined by $\mathcal{D}_q(j) = e^{-\zeta'_3(0)}$. A convenient quantity is the ratio, R_q , to the q th root of the full sphere determinant,

$$\begin{aligned} R_q(0) &= \frac{\mathcal{D}_q(0)}{(\mathcal{D}_1(0))^{1/q}} = \prod_{k=1}^{[(q-1)/2]} \left(\sin^2 \frac{\pi k}{q} \right)^{A_k(0)} \\ R_q(1) &= \frac{\mathcal{D}_q(1)}{(\mathcal{D}_1(1))^{1/q}} = \left(\frac{q}{\pi a} \right)^{2(1-1/q)} \prod_{k=1}^{[(q-1)/2]} \left(\sin^2 \frac{\pi k}{q} \right)^{A_k(1)} \end{aligned} \tag{18}$$

where the symmetry $A_{q-k}(j) = A_k(j)$ has been used to fold the product.

In any specific case these quantities can be computed. The first non-trivial q for which the formula applies is $q = 5$, ($\nu = 2, 3$). This is because for $q = 3$, the possible value of $\nu = 2$ is such that $\nu = 1 \pmod q$ and we are back to the one-sided (homogeneous or diagonal) case when a different method is needed. The method does not work for $q = 6$ so the next example is $q = 7$, ($\nu = 2, 3, 4, 5$).

With this in mind, the particular, ν -independent values of the A_k ,

$$\begin{aligned} A_1(0) &= 1 - \frac{1}{q}, & A_2(0) &= -\frac{4}{q}, & A_3(0) &= 1 - \frac{9}{q} \\ A_1(1) &= 0, & A_2(1) &= \frac{2(q-3)}{q} \end{aligned} \tag{19}$$

can be used to rewrite (18),

$$R_q(0) = \frac{(\sin^2 \frac{\pi}{q})^{(q-5)/q} (\sin^2 \frac{3\pi}{q})^{(q-9)/q} \prod_{k=4}^{[(q-1)/2]} \left(\sin^2 \frac{\pi k}{q} \right)^{A_k(0)}}{(4 \sec^2 \frac{\pi}{q})^{4/q}} \tag{20}$$

for $q > 5$ and

$$R_q(1) = \left(\frac{q}{\pi a}\right)^{2(1-1/q)} \left(\sin^4 \frac{2\pi}{q}\right)^{(q-3)/q} \prod_{k=3}^{[(q-1)/2]} \left(\sin^2 \frac{\pi k}{q}\right)^{A_k(1)} \quad (21)$$

for $q > 3$. The algebraic prefactor is the zero-mode effect.

The above expressions are useful numerically and I treat them, for now, purely in this light, as I do equation (15) for the A_k . The following section has some further analysis of the A_k quantities and an alternative route to the equations.

I give a few low values and plot a graph of $W = -\log R_{29}(0)$ against ν in the spin-0 case,

$$\begin{aligned} R_5(0) &= \left(\frac{47 - 21\sqrt{5}}{2}\right)^{1/5} \approx 0.463\,041\,35, & \nu &= 2, 3 \\ &\approx 0.368\,1520, & \nu &= 1, 4 \\ R_7(0) &\approx 0.321\,2271, & \nu &= 2, 3, 4, 5 \\ &\approx 0.167\,9911, & \nu &= 1, 6. \end{aligned}$$

The $\nu = 1$ and $\nu = 28$ values have been included for completeness. They were calculated from the expressions developed in section 5 or in [3].

4. Formal elaboration

In tune with my general policy of developing alternative, sometimes equivalent, approaches, I return to the starting form of the lens space ζ -function (3), which, now taking in definition (15), reads,

$$\zeta_3(s) = \zeta_3^{id}(s) + a^{2s} \sum_{n=1}^{\infty} \frac{A_n(j)}{n^{2s}}. \quad (22)$$

The second term, the non-identity part, is a Dirichlet series.

The A_n , are obviously periodic from (15),

$$A_{n+q}(j) = A_n(j), \quad (23)$$

and it is traditional in such cases either to Fourier transform or to break up the sum according to residue- q classes. I adopt the latter procedure and set $n = Nq + k$, where $N \in \mathbb{Z}$ and $0 \leq k < q$.

From (15),

$$A_0(0) = 0, \quad A_0(1) = 2/q - 2. \quad (24)$$

Since $A_n = A_k$ by (23), this means that k can be arranged to run from 1 to $q - 1$.

The non-identity ζ -functions, which are the major technical problem, then become, in standard fashion,

$$\begin{aligned} \zeta_3^{\text{nonid}}(s) &= a^{2s} A_0(j) \sum_{N=0}^{\infty} \frac{1}{(Nq)^{2s}} + a^{2s} \sum_{k=1}^{q-1} A_k(j) \sum_{N=0}^{\infty} \frac{1}{(Nq+k)^{2s}} \\ &= (a/q)^{2s} \left[A_0(j) \zeta_R(2s) + \sum_{k=1}^{q-1} A_k(j) \zeta_R(2s, k/q) \right] \end{aligned}$$

$$\begin{aligned}
 &= (a/q)^{2s} \left[A_0(j)\zeta_R(2s) + \sum_{k=1}^{(q-1)/2} A_k(j) (\zeta_R(2s, k/q) + \zeta_R(2s, 1 - k/q)) \right] \\
 &= (a/q)^{2s} \left[A_0(j)\zeta_R(2s) + \sum_{k=1}^{(q-1)/2} A_k(j) Z \left| \begin{matrix} k/q \\ 0 \end{matrix} \right| (2s) \right], \tag{25}
 \end{aligned}$$

which constitutes, perhaps, a neater expression than those in sections 3 and 4.

The reflection symmetry, from (15),

$$A_{q-k}(j) = A_k(j), \tag{26}$$

has been used in (25), and earlier.

From general principles, e.g. [11], there are no poles in the non-identity ζ -function, for fixed point free actions, Γ . From the behaviour at $s = 1/2$ of (25) this implies the sum rules,

$$\sum_{k=0}^{q-1} A_k(j) = 0, \tag{27}$$

which, together with (24) and $\zeta_R(0, w) = 1/2 - w$, yield, correctly, the values, $\zeta_3(0)$, of the total ζ -function as 0 and 1 for $j = 0$ and $j = 1$, respectively.

The local isometry of S^3 and S^3/Γ can be further exploited through the small-time expansion of the heat-kernel, the coefficients of which are integrals over local geometrical invariants. They are thus related simply by a volume $1/|\Gamma|$ factor and are determined by the identity ζ -functions, (8). The non-identity contributions must vanish. (Actually, in this case, only the first, volume, term in the expansion exists in any case.) The coefficients are proportional to $\zeta_3(-n)$, $n \in \mathbb{Z}$, and so, from (25), setting $\zeta_3^{\text{nonid}}(-n)$ to zero, there follow the sum rules:

$$\sum_{k=1}^{q-1} A_k(j) B_{2n+1}(k/q) = 0. \tag{28}$$

In fact, this is only a check because the expressions on the left vanish identically in view of the reflection symmetry, (26), of the A_k and that of the Bernoulli polynomials,

$$B_n(1 - x) = (-1)^n B_n(x).$$

There is, however, some valuable tactical information that can be extracted from (28). For example, set $n = 0$ and make use of (27). Then the sum rules reduce to the moments,

$$\sum_{k=0}^{q-1} k A_k(0) = 0, \quad \sum_{k=0}^{q-1} k A_k(1) = \frac{1}{2}(q - 1). \tag{29}$$

The fact that the coefficient of $1/(n/a)^{2s}$ in the ζ -function is a degeneracy, and hence integral, gives some general information about the A_n . For spin-0, the identity part of the ζ -function contributes a (non-periodic) factor of n^2/q to the (total) degeneracy, hence,

$$A_n(0) = D_n - \frac{n^2}{q}, \quad (D_n \in \mathbb{Z}), \quad \text{or} \quad q A_n(0) + n^2 = 0, \text{ mod } q. \tag{30}$$

The values (19) fit this pattern. The degeneracy, D_n , depends on q and the twisting, ν , generally. As a function of n , the image part, A_n , oscillates about zero and for large eigenvalues D_n is dominated by the n^2 term, which is the full sphere value divided by q , a volume factor¹.

¹ The n^2/q is the Weyl term. Using the sum rule, a sort of discrete eigenvalue counting function is $N_M = \sum_{n=1}^M D_n$, where $M = m(q - 1)$, giving $N_M \rightarrow M^3/3q$ as $M \rightarrow \infty$. M^2/a^2 is the eigenvalue, λ , and we see that for large M , $N_M \rightarrow \lambda^{3/2} |\mathcal{M}|/6\pi^2$, the Weyl asymptotic law. Such a crude argument works only for this leading behaviour.

As a further check, one can also evaluate the Casimir energy on the lens space Einstein universe, $T \times S^3/\Gamma$, this way. The values agree, as they must, with our earlier ones [5, 7]. Non-essential technical comments about this are therefore relegated to appendix B.

My main objective in this paper is the derivative, $\zeta'_3(0)$, and thence the determinant. This is easy to find by the approach of this section using Lerch's formula,

$$\zeta'_R(0, w) = \log(\Gamma(w)/\sqrt{2\pi}), \quad (31)$$

as in Ray's derivation of the torsion [2], or in the evaluation of the class number for quadratic forms of positive discriminant. Even better, one can transform the last expression in (25) to the image form by the Epstein functional equation and then use (12).

The answers are those given earlier, e.g. in (18). In showing this, one needs again the sums (27).

The main point I wish to make now is that, if q is odd prime, the determinant formulae in (18) can be written in terms of units, ϵ_k , of the q th cyclotomic number field defined by

$$\epsilon_k = \left(\frac{(1 - \zeta_q^k)(1 - \zeta_q^{-k})}{(1 - \zeta_q)(1 - \zeta_q^{-1})} \right)^{1/2} = \frac{\sin(\pi k/q)}{\sin(\pi/q)}, \quad k = 2, \dots, (q-1)/2, \quad (32)$$

where $\zeta_q = \exp(2\pi i/q)$ is a primitive q th root. See Hilbert's *Zahlbericht* [12], and also Borevich and Shafarevich [13] p 360. Franz [14] uses the square of ϵ .

Using (27) and (19), the results are

$$R_q(0) = \prod_{k=2}^{(q-1)/2} \epsilon_k^{2A_k(0)} \quad R_q(1) = \left(\frac{q \sin \pi/q}{\pi a} \right)^{2(1-1/q)} \prod_{k=2}^{(q-1)/2} \epsilon_k^{2A_k(1)}, \quad (33)$$

where $qA_k(j) \in \mathbb{Z}$.

The sum rule (27) plays a simple but vital role in the derivation of these formulae.

Because of the differing powers, A_k , these expressions are not invariant under all conjugations of the field.

It is a theorem that the product of units is a unit and also that a rational power of a unit is a unit, e.g. [15], §§ 90, 91, 105. Hence, formally, the ratios in (33) are cyclotomic units (up to a factor for spin-1).

I remark that the corresponding spin-1/2 formula is

$$R_q(1/2) = \prod_{k=1,3,\dots}^{2q-1} \bar{\epsilon}_k^{A_k(1/2)}, \quad (34)$$

where $\bar{\epsilon}_k$ are $2q$ th cyclotomic units, and the $A_k(1/2)$ are the (partial) degeneracies arising from the non-identity actions. Their form is given in [7] and again we have,

$$\sum_{k=1,3,\dots}^{2q-1} A_k(1/2) = 0,$$

corresponding to $\zeta_3(0) = 0$ for massless spin half.

5. Return to the homogeneous case

In [3], the homogeneous determinants were computed using an expression for the lens space ζ -function in terms of that on the orbifolded two sphere. In this paper, for variety, I wish to

take the spin-0, one-sided Epstein form [5],

$$\zeta_3(s) = -\frac{a^{2s}}{2|\Gamma|} \sum_{\gamma} \frac{1}{\sin \theta_{\gamma}} \frac{\partial}{\partial \theta_{\gamma}} Z \left| \begin{matrix} 0 \\ \theta_{\gamma}/2\pi \end{matrix} \right| (2s), \tag{35}$$

further to obtain an alternative expression.

There may be a certain amount of overkill in this, but I have found it helpful to have a number of equivalent expressions to hand, if only for numerical peace of mind. In view of the many identities and relations for the ζ -functions, Bernoulli polynomials etc, one should expect several versions of the same thing.

As before, the image form is [5]

$$\zeta_3(s) = -\frac{a^{2s}}{2|\Gamma|} \pi^{2s-1/2} \frac{\Gamma(1/2-s)}{\Gamma(s)} \sum_{\gamma} \frac{1}{\sin \theta_{\gamma}} \frac{\partial}{\partial \theta_{\gamma}} Z \left| \begin{matrix} \theta_{\gamma}/2\pi \\ 0 \end{matrix} \right| (1-2s), \tag{36}$$

which can be re-expressed in terms of the more familiar Hurwitz ζ -function,

$$\zeta_3(s) = \frac{\pi^{2s-3/2}}{2|\Gamma|} \frac{\Gamma(3/2-s)}{\Gamma(s)} \sum_{\gamma} \frac{1}{\sin \theta_{\gamma}} \left(\zeta_R \left(2-2s, \frac{\theta_{\gamma}}{2\pi} \right) - \zeta_R \left(2-2s, 1 - \frac{\theta_{\gamma}}{2\pi} \right) \right). \tag{37}$$

This general formula can be used to calculate any quantity required. There is, however, a slight awkwardness whenever $\theta_{\gamma} = \pi$, since both numerator and denominator in the summand vanish. This happens in even lens spaces in which case this term, as well as the identity one, is treated separately. This can be done by extracting the \mathbb{Z}_2 ζ -function scaled by a volume factor.

Hence, the formula suitable for even q is

$$\begin{aligned} \zeta_3(s) &= a^{2s} \frac{2}{q} (1 - 2^{2-2s}) \zeta_R(2s-2) + \frac{\pi^{2s-3/2}}{q} \frac{\Gamma(3/2-s)}{\Gamma(s)} \\ &\times \sum_{p=1}^{q/2-1} \frac{1}{\sin(2\pi p/q)} \left(\zeta_R \left(2-2s, \frac{p}{q} \right) - \zeta_R \left(2-2s, 1 - \frac{p}{q} \right) \right). \end{aligned} \tag{38}$$

Further manipulations would take us to the expressions in [3, 16].

Sometimes it is not necessary to make this step. For example, when evaluating the Casimir energy this way, by setting $s = -1/2$ in (37), the Hurwitz functions are evaluated at an odd argument and combine to a sum that gives a trigonometric result which cancels against the $\sin \theta$ on the bottom. The previous answers in terms of sums of cosecants [3, 5] can be obtained in this very roundabout, dual fashion. I offer some further remarks in appendix A.

For completeness, the odd q formula is

$$\begin{aligned} \zeta_3(s) &= \frac{a^{2s}}{q} \zeta_R(2s-2) + \frac{\pi^{2s-3/2}}{q} \frac{\Gamma(3/2-s)}{\Gamma(s)} \\ &\times \sum_{p=1}^{(q-1)/2} \frac{1}{\sin(2\pi p/q)} \left(\zeta_R \left(2-2s, \frac{p}{q} \right) - \zeta_R \left(2-2s, 1 - \frac{p}{q} \right) \right). \end{aligned} \tag{39}$$

The derivatives at 0, $\zeta_3'(0)$, follows more or less as before,

$$\zeta_3'(0) = -\frac{a^{2s}}{2\pi^2 q} \zeta_R(3) + \frac{1}{2\pi q} \sum_{p=1}^{(q-1)/2} \frac{1}{\sin(2\pi p/q)} \left(\zeta_R \left(2, \frac{p}{q} \right) - \zeta_R \left(2, 1 - \frac{p}{q} \right) \right), \tag{40}$$

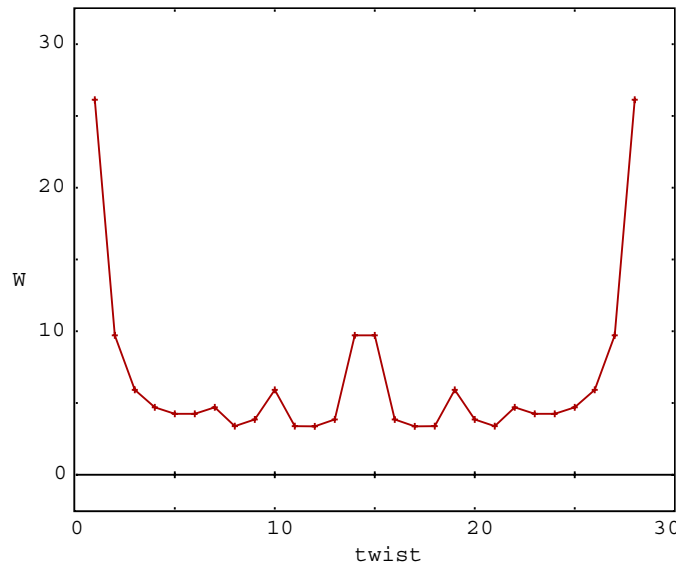


Figure 1. $W = -\log R_{29}$ for conformal scalars on a lens space of order 29 for twistings ν or, equivalently, λ , from 1 to 28, ($\lambda\nu = 1 \pmod{29}$).

for odd q and, for even,

$$\zeta'_3(0) = 3 \frac{a^{2s}}{\pi^2 q} \zeta_R(3) + \frac{1}{2\pi q} \sum_{p=1}^{q/2-1} \frac{1}{\sin(2\pi p/q)} \left(\zeta_R\left(2, \frac{p}{q}\right) - \zeta_R\left(2, 1 - \frac{p}{q}\right) \right), \quad (41)$$

which constitute decent, numerically amenable forms.

6. Discussion

The main results of this short paper are the explicit expressions (18) and (33). These yield the determinants on the lens space, $L(q; 1, \nu)$, with a restriction on the coprime q and ν which can be met, sufficiently, by restricting q to be odd prime.

Figure 1 graphically summarizes the numerics. It obviously exhibits the lens space homeomorphism $L(q; 1, \lambda) \sim L(q; 1, \lambda')$ when $\lambda' = \pm\lambda \pmod{q}$ and the homeomorphism when $\lambda'\lambda = \pm 1 \pmod{q}$ can also be observed, e.g., $L(29; 1, 8) \sim L(29; 1, 11)$. These equalities explain the appearance of the graph as a series of extrema and therefore account, partly, for the startling similarity with the plot of the corresponding Casimir energy in [7].

The fact that the image contribution to the determinant is a cyclotomic unit is presented simply as an amusing formal identity.

Appendix A

I give some further analysis relevant for the homogeneous lens space conformal ζ -function, another method of obtaining which is to apply Plana summation directly to the sum definition,

$$\zeta_3(s, m) = \sum_{\Gamma} \sum_{n=1}^{\infty} \frac{n \sin n\theta_{\gamma}}{\sin \theta_{\gamma} (n^2 + m^2)^s},$$

where, because I can, I have added an extra mass-like term. Basic contour manipulations, cf [17], lead straightforwardly to the ‘real’ integral for the non-identity, ‘image’, part of the ζ -function, in the range $s < 1$,

$$\zeta_3^{\text{nonid}}(s, m) = 4a^{2s} \sin \pi s \sum_{\Gamma} \int_m^{\infty} \frac{y \, dy}{(y^2 - m^2)^s} \frac{\sinh y(\theta_{\gamma} - \pi)}{\sin \theta_{\gamma} \sinh \pi y}, \quad 0 < \theta_{\gamma} < 2\pi.$$

I now set $m = 0$ when the integral, [18, 19], 3.524.1, yields (37) which can now be extended to all s . Furthermore, [19] 3.524.10–16, enable one to obtain trigonometric formulae when $s = -1/2$, etc. This is yet another longwinded way of deriving the cosecant sums in [3, 5].

Useful general formulae to bear in mind are

$$\int_0^{\infty} dy \, y^{2r} \frac{\sinh y(\theta - \pi)}{\sinh \pi y} = -\frac{1}{2} \frac{d^{2r}}{d\theta^{2r}} \cot \frac{\theta}{2} \tag{A.1}$$

and

$$\int_0^{\infty} dy \, y^{2r+1} \frac{\sinh y(\theta - \pi)}{\sinh \pi y} = \frac{1}{2} \frac{d^{2r}}{d\theta^{2r}} \operatorname{cosec} \frac{\theta}{2}. \tag{A.2}$$

Dividing by $\sin \theta$, the right-hand side of (A.1) is a polynomial in $\operatorname{cosec}^2 \theta/2$ which can be determined by recursion².

Expressed in terms of the series forms, these equations are equivalent to those of Eisenstein [21] (see, e.g., Hancock [22] p 32 ex 5). Defining,

$$(g, x) = \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^g}, \quad g \in \mathbb{Z},$$

one has the connections,

$$(2g, x) = Z \left| \begin{matrix} x \\ 0 \end{matrix} \right| (2g), \quad (2g+1, x) = -\frac{1}{2g} \frac{\partial}{\partial x} Z \left| \begin{matrix} x \\ 0 \end{matrix} \right| (2g).$$

The polynomial referred to above is

$$(2g, x) = \pi^{2g} \sum_{k=1}^g (-1)^{k+1} A_{2g,2k} \operatorname{cosec}^{2k} \pi x,$$

with the recurrence relation

$$A_{2g+2,2k} = \frac{1}{2g(2g+1)} ((2k-1)(2k-2)A_{2g,2k-2} + 4k^2 A_{2g,2k}),$$

and $A_{2g,2g} = 1$. Eisenstein remarks that the coefficients are related simply to Bernoulli numbers.

Likewise one has the expression

$$(2g+1, x) = \pi^{2g+1} \cos \pi x \sum_{k=1}^g (-1)^{k+1} A_{2g+1,2k+1} \operatorname{cosec}^{2k+1} \pi x.$$

If there are further operations to be performed, such as summing over the angles, then these polynomials are not necessarily the best way of proceeding. Leaving things as series is sometimes more effective [6, 23].

² As a point of historical interest, the quoted integrals in [19] are all taken from the classic compilation by Bierens de Haan [20]. No references are given in this work, the author referring for these to vols IV, V and VIII of the Mémoires of the Royal Dutch Academy. Only vol IV is available to me. The lists in these volumes are, apparently, even more extensive than the mammoth [20]!

Appendix B

I elaborate on the computation of the Casimir energy using the ζ -function form (22), and make some relational comments on earlier calculations.

The non-identity contribution is

$$\begin{aligned}
 \frac{1}{2}\zeta_3^{\text{nonid}}(-1/2) &= \frac{q}{2a} \left[A_0(j)\zeta_R(-1) + \sum_{k=1}^{(q-1)/2} A_k(j) \left(\zeta_R\left(-1, \frac{k}{q}\right) + \zeta_R\left(-1, 1 - \frac{k}{q}\right) \right) \right] \\
 &= -\frac{q}{4a} \left[A_0(j)B_2 + \sum_{k=1}^{(q-1)/2} A_k(j) \left(B_2\left(\frac{k}{q}\right) + B_2\left(1 - \frac{k}{q}\right) \right) \right] \\
 &= -\frac{q}{4a} \left[A_0(j)B_2 + 2 \sum_{k=1}^{(q-1)/2} A_k(j)B_2\left(\frac{k}{q}\right) \right] \\
 &= -\frac{q}{2a} \left[\frac{A_0(j)}{12} + \sum_{k=1}^{(q-1)/2} A_k(j) \left(\frac{k^2}{q^2} - \frac{k}{q} + \frac{1}{6} \right) \right], \tag{B.1}
 \end{aligned}$$

in terms of Bernoulli polynomials.

For simplicity, I carry only the spin-0 results further by using the vanishing moments (29) to give,

$$\frac{1}{2}\zeta_3^{\text{nonid}}(-1/2) = -\frac{1}{4qa} \sum_{k=0}^{q-1} k^2 A_k(0). \tag{B.2}$$

Evaluation of the A_k purely numerically from (15) yields, for (B.1), or (B.2), values in complete agreement with those computed in [7] which were there expressed in terms of generalized Dedekind sums. By summing over n first [5, 7], one comes quickly to the angle sum form,

$$\frac{1}{2}\zeta_3^{\text{nonid}}(-1/2) = -\frac{1}{16qa} \sum_{p=1}^{q-1} \operatorname{cosec}^2 \pi p/q \operatorname{cosec}^2 \pi p\nu/q, \tag{B.3}$$

which was determined in [7] by means of a recursion technique as an *explicit* quartic polynomial in q whose coefficients were found as numbers, for any given ν .

Arising from the same ingredients, the angle form (B.3) must be derivable from the moment form (B.2). Using the finite Fourier transforms,

$$\begin{aligned}
 \sum_{k=0}^{q-1} \cos \frac{2\pi k p \nu}{q} &= 0, & \sum_{k=0}^{q-1} k \cos \frac{2\pi k p \nu}{q} &= -\frac{q}{2} \\
 \sum_{k=0}^{q-1} k^2 \cos \frac{2\pi k p \nu}{q} &= \frac{q}{2} \operatorname{cosec}^2 \frac{\pi p \nu}{q} - \frac{q^2}{2}, \tag{B.4}
 \end{aligned}$$

on (15), with some trigonometry the desired equivalence can be shown. These transforms also confirm the properties of the A_k , (27) and (29).

At the moment I do not have a means of finding the general form for the partial degeneracies, A_k , and so the present way of calculating the Casimir energy is no better, apart from novelty, than simply performing the sum (B.3) numerically for given q and ν . Furthermore, in the present procedure, q has been assumed prime, for convenience.

For $j = 0$, the denominator in (15) can be divided into the numerator at the expense of introducing a product,

$$A_k(0) = 2^{k-1} \frac{1}{q} \sum_{p=1}^{q-1} \prod_{r=1}^{k-1} \left(\cos \frac{2\pi p}{q} - \cos \left(\frac{2\pi p r}{q} + \frac{2\pi r}{k} \right) \right),$$

but there seems to be no advantage in this.

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